Lecture 3 : Sheaf cohomology

X scheme of finite type over $k = \bar{k}$

sheaf cohomology is a functor

$$\mathrm{QCoh}(X) \to \mathrm{GrVect}_k \quad (\mathrm{Gr}\,\mathrm{QCoh}(X))$$

$$\mathcal{F} \mapsto (H^i(X, \mathcal{F}))_{i \in \mathbb{Z}}$$

s.t. (1) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$

(2) for every short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

have functorial l.e.s:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to$$
$$\to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'') \to \cdots$$

(3) for every morphism $\pi : X \to Y$ there is a natural homomorphism

$$H^*(Y, \pi_* \mathcal{F}) \to H^*(X, \mathcal{F})$$

which is an isomorphism if π is affine.

(4) $H^i(X, \operatorname{colim} \mathcal{F}_j) = \operatorname{colim} H^i(X, \mathcal{F}_j)$

(5) X proper$/k$ $\Rightarrow$ $H^i(X, \mathcal{F})$ fin. dim'l

(6) X (quasi) projective$/k$ $\Rightarrow H^i(X, \mathcal{F}) = 0$ for all $i > \dim(X)$

(3) $\Rightarrow$ affine schemes

(7) X, Y var$/k$. $H^i(X \times Y, pr_X^* \mathcal{F} \otimes pr_Y^* \mathcal{G}) = \bigoplus_{p+q=i} H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G})$
(Künneth)

Čech cohomology

$X = \bigcup_{i \in I} U_i$ open (affine) cover $\mathcal{U}$

Čech complex $(\check{C}^*(\mathcal{U}, \mathcal{F}), \check{d})$

degree n v.s.: $\check{C}^n(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \dots < i_n} \mathcal{F}(\underbrace{U_{i_0} \cap \dots \cap U_{i_n}}_{U_{i_0 \dots i_n}})$

differential

$$\check{d}^n : \check{C}^n(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$$

$$(\alpha_{j_0, \dots, j_n}) \longmapsto \Big(\sum_{k=0}^{n+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{n+1}} \big|_{U_{i_0 \dots i_{n+1}}} \Big)$$

$$\check{H}^n(\mathcal{U}, \mathcal{F}) := H^n(\check{C}^*(\mathcal{U}, \mathcal{F}))$$

Theorem X noetherian, separated $/k$.
$\mathcal{U}$ affine open cover of X
$\check{H}^*(\mathcal{U}, -) \cong H^*(X, -)$
as δ-functors.

~~X noetherian separated~~

(Hard) Exercise: prove the desiderata
(1) – (6) from this definition

Cohomology of projective space

Recall Serre twists $k[x_0,\dots,x_n](m) = \bigoplus_{s\in\mathbb{Z}} k[x_0,\dots,x_n]_{s-m}$

graded modules over $k[x_0,\dots,x_n]$

$\rightsquigarrow$ defines a quasicoherent sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$ which is a line bundle:

$$\mathcal{O}_{\mathbb{P}^n}(m)\big|_{U_i} = x_i^{-m}\cdot \mathcal{O}_{U_i}$$

with transition functions

$$\mathcal{O}(m)(U_{ij}) \longrightarrow \mathcal{O}(m)(U_{ji})$$
$$f \longmapsto \left(\frac{x_i}{x_j}\right)^m f$$

Properties: 1) $\mathcal{O}(m)\otimes\mathcal{O}(n) \cong \mathcal{O}(m+n)$

2) $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\,\mathcal{O}(1)$

<u>Theorem</u> 1) $\exists$ iso of graded rings

$$k[x_0,\dots,x_n] \to \bigoplus_{m\in\mathbb{Z}} H^0(\mathbb{P}^n,\mathcal{O}(m))$$

2) $H^n(\mathbb{P}^n,\mathcal{O}(-n-1)) \cong k$

3) $H^i(\mathbb{P}^r,\mathcal{O}(m)) = 0 \quad 0<i<r$ for all $m\in\mathbb{Z}$.

4) $H^0(\mathbb{P}^r,\mathcal{O}(m)) \times H^r(\mathbb{P}^r,\mathcal{O}(-m-n-1)) \to k$
$\exists$ perfect pairing.

<u>Proof</u> Čech cohomology computation using the standard affine open cover of $\mathbb{P}^n$.

$$\mathbb{P}^n = U_0 \cup \dots \cup U_n$$
$$U_i = \{x_i \neq 0\}$$

Notation:

$$S = k[x_0, \dots, x_n]$$

$$S_{x_{i_0} \cdots x_{i_p}} = k[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}}$$

$\frac{x_u}{x_i}$ arbitrary degree

$$\Gamma(U_i, \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m)) = \bigoplus_{m \in \mathbb{Z}} x_i^{-m} k[x_0, \dots, \hat{x}_i, \dots, x_n]$$ degree m

$$= S_{x_i}$$

$\Gamma(U_i, \mathcal{O}(m)) = (k[x_0, \dots, x_n](m)_{x_i})_0$

$=$ degree 0 part of

$k[x_0, \dots, x_n](m)_{x_i}$

Beginning of the Čech complex

degree 0 $\qquad$ degree 1

$$\bigoplus_{i=0}^{n} S_{x_i} \xrightarrow{\check{d}^0} \bigoplus_{i<j} S_{x_i x_j}$$

$$\check{d}^0(\alpha_i) = (\alpha_i - \alpha_j)_{i<j}$$

$$\check{d}^0(\alpha) = 0 \Leftrightarrow \alpha_i - \alpha_j = 0 \quad \forall i<j$$

$$\Rightarrow \alpha_i = \alpha_j \in S_{x_i} \cap S_{x_j} \quad \forall i, j$$

$$\Rightarrow \alpha_i \in \bigcap_{i=0}^{n} S_{x_i}$$

$$\Rightarrow \alpha \equiv \alpha_i \in S$$

$$\Rightarrow H^0(\mathbb{P}^n, \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m)) \cong S \quad \rightarrow (1)$$

$$\Rightarrow \dim_k H^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{m+n}{m}$$ for $m \geq 0$

End of Čech complex

$$\bigoplus_{i=0}^{n} S_{x_0 \cdots \hat{x}_i \cdots x_n} \xrightarrow{\check{d}^n} S_{x_0 \cdots x_n}$$

$$\operatorname{im}(\check{d}_n) = \langle x_0^{l_0} \cdots x_n^{l_n} \mid \exists i \text{ s.t. } l_i \geq 0 \rangle$$

$$\Rightarrow \quad \check H^n\Big(\mathcal U, \bigoplus_{m\in\mathbb Z} \mathcal O(m)\Big) = S_{x_0\cdots x_r}/\mathrm{im}(\check d^r)$$

$$= \langle x_0^{l_1}\cdots x_n^{l_n} \mid \forall i \ \ l_i<0\rangle$$

$$= \bigoplus_{m\in\mathbb Z} \langle x_0^{l_1}\cdots x_n^{l_n} \mid \forall i \ \ l_i<0 \text{ and } \textstyle\sum l_i = m\rangle$$

$$\Rightarrow \quad \check H^n(\mathcal U, \mathcal O(-n-1)) \cong k \qquad (2)$$

$$\check H^n(\mathcal U, \mathcal O(m)) \cong 0 \qquad \text{for } m>-n-1$$

$$\dim_k \check H^n(\mathcal U, \mathcal O(m)) = \binom{-m-1}{-n-m-1} \qquad m \leq -n-1.$$

hyperplane $Z_0 = \{x_0=0\} \subset \mathbb P^n$, $Z_0 \cong \mathbb P^{n-1}$

⇝ SES
$$0 \to \mathcal I_{Z_0} \xrightarrow{\cdot x_0} \mathcal O_{\mathbb P^n} \to \mathcal O_{Z_0} \to 0$$
$$\mathcal I_{Z_0} \cong \mathcal O(-1), \qquad \mathcal O_{Z_0} \cong \mathcal O_{\mathbb P^{n-1}}$$

$\downarrow \otimes \mathcal O(m)$

$$0 \to \mathcal O(m-1) \xrightarrow{\cdot x_0} \mathcal O_{\mathbb P^n}(m) \to \mathcal O_{\mathbb P^{n-1}}(m) \to 0$$

intersecting hyperplanes generally a hyperplane in the smaller

plan: prove (3) by induction on n, m

~~LES [illegible]~~

$$0 \to H^0(\mathbb P^n, \mathcal O(m-1)) \to H^0(\mathbb P^n, \mathcal O_{\mathbb P^n}(m)) \to H^0(\mathbb P^{n-1}, \mathcal O_{\mathbb P^{n-1}}(m)) \to 0$$

SES Pascal's rule:
$$\binom{n+m}{m} = \binom{n+m-1}{m} + \binom{n+m-1}{m-1}$$

similarly ~~[illegible]~~

(2)

$$0 \to H^{n-1}(\mathbb P^{n-1}, \mathcal O_{\mathbb P^{n-1}}(m))$$

~~LES [illegible]~~

$$\to H^n(\mathbb P^n, \mathcal O(m-1)) \to H^n(\mathbb P^n, \mathcal O(m)) \to 0$$

Remaining: $i = 1, n-1$

$\Rightarrow \check{H}^i_{\mathcal{U}}(\mathbb{P}^n, \mathcal{O}(m-1)) \xrightarrow{\cdot x_0} \check{H}^i_{\mathcal{U}}(\mathbb{P}^n, \mathcal{O}(m))$ is a

bijection for all $1 \leq i \leq n$

(use LES for $1 < i < n-1$.

$\check{C}(\mathcal{U}, \bigoplus_m \mathcal{O}(m))_{x_0}$ is exact in degrees > 0

~~$\Rightarrow x_0 : H^i(\mathbb{P}^n, \mathcal{O}(m-1))_{x_0}$~~

$\Rightarrow x_0 : H^i(\mathbb{P}^n, \bigoplus_m \mathcal{O}(m)) \xrightarrow{x_0} H^i(\mathbb{P}^n, \bigoplus_m \mathcal{O}(m+1))$

nilpotent. bijection for $0 < i < n$

$\Rightarrow H^i(\mathbb{P}^n, \bigoplus_m \mathcal{O}(m)) = 0$.

(4) $H^0(\mathbb{P}^n, \mathcal{O}(m)) \times H^n(\mathbb{P}^n, \mathcal{O}(-m-n-1)) \to H^n(\mathcal{O}(-n$

$= \langle x_0^{a_0} \cdots x_n^{a_n} \mid \sum a_i = m, \; a_i \geq 0 \rangle \times \langle x_0^{b_0} \cdots x_n^{b_n} \mid \sum b_i = -m-n-1, \; b_i < 0 \rangle$

$\xrightarrow[\text{multiply}]{} \langle x_0^{c_0} \cdots x_n^{c_n} \mid \sum c_i = -n-1, \; c_i < 0 \rangle$ $\square$

Canonical divisor

X ~~smooth p~~ variety ~~proj.~~ $/k = \bar{k}$

$\Omega_{X/k}$: sheaf of Kähler differentials
~~for~~

$$\Omega^1_{X/k}(\mathrm{Spec}(A)) = \left(\bigoplus_{a \in A} A\,da\right) \Big/ \left\langle \begin{array}{l} d(a+a') = da + da' \\ d(ab) = (da)b + a\,db \\ d(\lambda) = 0 \quad \forall \lambda \in k \end{array} \right\rangle$$

$\Downarrow$

$$\Omega^1_{X/k}\left(\mathrm{Spec}\left(k[x_1,\dots,x_n]/(f_1,\dots,f_m)\right)\right) = \bigoplus_{i=1}^{n} A\,dx_i \Big/ \left(\frac{\partial f_j}{\partial x_i}\right)$$

X ns $\overset{\text{Jac. Crit.}}{\Longleftrightarrow}$ $\Omega^1_{X/k}$ locally free sheaf of rank $\dim(X)$

~~$\boxed{X \text{ ns.}}$ $\omega_X := \bigwedge \det$~~

Def X ns variety $/k = \bar{k}$

$$\omega_X = \det(\Omega^1_X) = \bigwedge^{\dim(X)} \Omega^1_X$$

<u>canonical line bundle</u>

A divisor K_X with $\omega_X \cong \mathcal{O}_X(K_X)$ is called a <u>canonical divisor</u> of X

Example $X = \mathbb{P}^n$ have Euler sequence.

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

$$\Rightarrow \det(\Omega^1_{\mathbb{P}^n}) \otimes \det(\mathcal{O}_{\mathbb{P}^n}) = \det(\mathcal{O}(-1)^{\oplus n+1})$$
$$= \mathcal{O}(-1)^{\otimes n+1} = \mathcal{O}(-n-1)$$

$\Rightarrow$ have perfect pairing.

$$H^i(\mathbb{P}^n, \mathcal{O}(m)) \times H^{n-i}(\mathbb{P}^n, \underbrace{\mathcal{O}(-m)}_{\mathcal{O}(m)^\vee} \otimes \omega_{\mathbb{P}^n}) \to k.$$

$$\Rightarrow H^i(\mathbb{P}^n, \mathcal{O}(m)) \cong H^{n-i}(\mathbb{P}^n, \mathcal{O}(m)^\vee \otimes \omega_{\mathbb{P}^n})$$

Theorem (Serre duality)

X ns connected projective / $k = \bar{k}$ variety.

$\mathcal{F}$ locally free sheaf on X

$\exists$ perfect pairing $H^i(X, \mathcal{F}) \times H^{\dim(X)-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to k.$

in particular

$$H^i(X, \mathcal{F}) \cong H^{\dim(X)-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$$

e.g. Serre duality for curves:

X ns connected proj. variety dim 1 $/ k = \bar{k}$

$\mathcal{L} \in \mathrm{Pic}(X)$

$$H^0(X, \mathcal{L}) \cong H^1(X, \mathcal{L}^\vee \otimes \Omega^1_X)^\vee$$